where

$$(\rho c)_{i,j} = \left\{ \int_0^{T_{i,j}} \rho c dT \right\} / T_{i,j}$$

and the heat flux from Eq. (5) can be written in numerical form as

$$q_n(j\Delta t) = \{Q_n[(j+1)\Delta t] - Q_n[(j-1)\Delta t]\}/2\Delta t \quad (7)$$

Thus by this method the heat flux can be numerically evaluated for arbitrary $T(0,t)$ and any prescribed variation of thermal properties with temperature. Additional details and computational techniques are discussed in Ref. 6.

To illustrate the accuracy of the present method, numerical solutions will be presented for two special cases of property variation; α independent of ϕ with variable ρc and k and constant ρc with variable k and α. When α is independent of ϕ, Eq. (2) is linear and of the same form as the corresponding equation obtained in terms of T when properties are assumed constant. Thus Eq. (1) obtained for constant properties is applicable when $\alpha(\phi)$ = constant if T in that equation is replaced by ϕ. Table 1 presents for $\alpha(\phi)$ = constant a comparison of theoretical solutions and numerical solutions obtained through applications of Eq. (1) and Eq. (7) for two boundary conditions for which exact solutions for heat flux can be readily obtained. For each boundary condition it is noted that the error associated with each numerical method is small, and decreases with increasing N.

The case of ρc constant with variable k and α was considered, since for this case an exact solution to the governing equations obtained by Yang[7] for a semi-infinite solid and a stepwise change in surface temperature is available for comparison purposes. Figure 1 presents a comparison of theoretical and numerically determined temperature profiles in terms of Yang's similarity variables T/T_s and $x/2(k_o\Delta t/\rho c)^{1/2}$ for Yang's case of linear variation of k with temperature as noted in the figure. Since for the present case ρc is assumed constant, α also varies linearly with temperature. Table 2 presents a comparison of numerically determined and exact heat-transfer rates based on Yang's solution. In the computations for Fig. 1 and Table 2, α varied by about $\pm$ 50%. It is noted from Fig. 1 and Table 2 that the agreement between Yang's solution and the present solution is very good and increases with N. The results in Tables 1 and 2 and Fig. 1 are considered representative of the accuracy in the computation of heat-transfer rates that can be obtained for arbitrary $T(0,t)$ and variable substrate properties using the present method. If α for the heat-flux gage substrate is independent of ϕ or if it is established by the present method that the influence of α is small, Eq. (1) with T replaced by ϕ is preferred over the present method since numerical evaluation of Eq. (1) requires less computer time.

Hartunian and Varwig[8] analyzed the influence of variable substrate properties on heat flux inferred from surface temperature measurements made by thin-film heat-flux gages. Code 7740 pyrex, one of the substrate materials considered in Ref. 8, is a commonly used substrate for thin-film heat-flux gages. Hence it is of interest to analyze by the present method the influence of variable properties in this material on heat-transfer rates inferred from surface temperature histories. Figure 2 shows results of computations by the present method for two boundary conditions, $\phi \propto t^{1/2}$ and ϕ = const, for 7740 pyrex using property values from Table II, Ref. 8. Here, q_v/q_o is the ratio of the heat flux computed considering all properties variable to the heat flux computed assuming constant properties, whereas q_α/q_o is the ratio of heat flux computed assuming α independent of ϕ to the heat flux computed assuming constant properties. It is noted in Fig. 2 first that the boundary condition imposed has an influence on the ratios and second, that it is important to consider for this material the variation in α as well as k when temperature change is large. The latter is evidenced by the departure of q_α/q_o from q_v/q_o as both depart from unity with increasing temperature for both boundary conditions. The curves in Fig. 2 indicate that for 7740 pyrex neither the assumption of constant properties nor the assumption of α independent of ϕ is satisfactory at higher values of temperature change. The results in Fig. 2 differ somewhat from those obtained by Hartunian and Varwig for the same material using an approximate theoretical solution of the governing equations and boundary conditions equivalent to those in Fig. 2. For example, Hartunian and Varwig list values of 1.45 and 1.44 for q_α/q_o and q_v/q_o at T = 150°C for both boundary conditions. However, the present method yields corresponding values of 1.52 and 1.35 for the parabolic boundary condition and 1.41 and 1.30 for the step-boundary condition. In view of the verification obtained for the present method it is concluded that for 7740 pyrex the influence of both the boundary condition and variable diffusivity are each more significant than suggested by Hartunian and Varwig.

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Center of Flexure of Arbitrary Cross Sections

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In a previous paper[1] the general formula for the position of the center of flexure has been developed in the form

$$y_0 = \frac{1}{F} \iint \left\{ y \frac{\partial \psi}{\partial y} - x \frac{\partial \psi}{\partial x} - F \frac{yx^2}{2I} \right\} dxdy \quad (1)$$

where $Oxyz$ = orthogonal coordinate system; x,y = principal axes of the cross section; F = vertical load parallel to the x axis; $I = \iint x^2 dxdy$ = second moment of area about y axis; $\psi(x,y)$ = stress function defined by

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x^2} = \frac{\nu}{1+\nu} \frac{Fy}{I}$$

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and

$$\psi = \int (Fx^2/2I)dy$$

at the boundary; and ν = Poisson's ratio.

For the purpose of practical computation, we transform the formula (1) to obtain

$$y_0 = \frac{1}{F}\left\{\int |y\psi|_{y_1}^{y_2}dx - \int |x\psi|_{x_1}^{x_2}dy - \frac{F}{2I}\iint yx^2dxdy\right\} \quad (2)$$

The surface integral in Eq. (1) is herewith transformed to a line integral dependent only upon the boundary conditions. This transformation simplifies considerably the determination of the center of flexure and gives also an independent and simpler proof that the position of the center of flexure does not depend on Poisson's ratio.[1]

To apply the formula (2) we consider an equilateral triangular cross section in Fig. 1 where the position of the center of flexure has already been determined in Ref. 2 by another method.

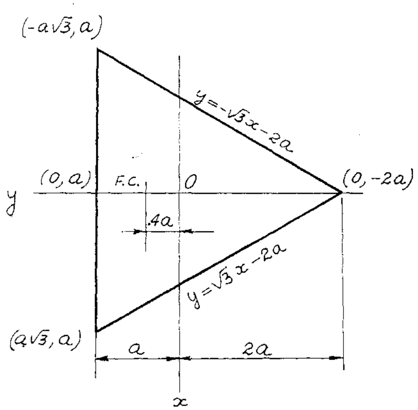

Fig. 1 Equilateral triangular cross section.

Evaluating the integrals in Eq. (2) using the values at the boundary

$$\psi_{\text{BOUND.}} = \frac{F}{2I}\int x^2dy = \frac{F}{2I} \times \left(\frac{y^3}{9} + \frac{2}{3}ay^2 + \frac{4}{3}a^2y - \frac{19}{9}a^3\right) = \frac{F}{2I}\left(\pm(3)^{1/2}\frac{x^3}{3} - 3a^3\right) \quad (3)$$

we obtain

$$\int |y\psi|_{y_1}^{y_2}dx = \frac{F}{I}\int_{a\,(3)^{1/2}}^{0}[(3)^{1/2}x - 2a] \times \left[(3)^{1/2}\frac{x^3}{3} - 3a^3\right]dx = -\frac{9(3)^{1/2}}{5}\frac{Fa^5}{I} \quad (4)$$

$$\int |x\psi|_{x_1}^{x_2}dy = \frac{F}{I}\int_{-2a}^{a}(y + 2a) \times \left(\frac{y^3}{9} + \frac{2}{3}ay^2 + \frac{4}{3}a^2y - \frac{19}{9}a^3\right)dy = -\frac{27(3)^{1/2}}{10}\frac{Fa^5}{I} \quad (5)$$

$$\frac{F}{2I}\iint yx^2dxdy = \frac{F}{I}\int_0^{a(3)^{1/2}} x^2\left(\int_{(3)^{1/2}x-2a}^{a} ydy\right)dx = \frac{3}{10}(3)^{1/2}\frac{Fa^5}{I} \quad (6)$$

Since

$$I = \iint x^2dxdy = \tfrac{3}{2}(3)^{1/2}a^4 \quad (7)$$

using Eqs. (4–6) we obtain from (2)

$$y_0 = \left(-\frac{9(3)^{1/2}}{5} + \frac{27(3)^{1/2}}{10} - \frac{3(3)^{1/2}}{10}\right)\frac{2a}{3(3)^{1/2}} = \frac{2}{5}a \quad (8)$$

This confirms the previous result.[2]

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